Square root process/law

Square root process is any stochastic process defined by the equation: $dX(t) = \sigma_t 2\sqrt{|X(t)|}dW(t) + \mu_t dt$, where σ_t is deterministic function and μ_t an adapted to the Brownian filtration process.

We shall analyse particular cases.

The easiest example is $X(t) = B^2(t)$ where $B(t)$ is standard Brownian Motion. In this case

$$W(t) = \int_0^t \mathrm{sgn} B(s) dB(s),$$

where

$$\mathrm{sgn}\, x = \begin{cases} -1 & \text{if} \quad x \le 0 \\ 1 & \text{if} \quad x > 0, \end{cases}$$

and we have

$$dX(t) = 2\sqrt{X(t)}dW(t) + 1dt, \quad X(0) = x > 0. \tag{1}$$

More general square root process called squared Bessel process of dimension $\delta \ge 0$, and denoted $BESQ^\delta(x)$, is defined as the unique strong solution of

$$dX(t) = 2\sqrt{X(t)}dW(t) + \delta t, \; X(0) = x \ge 0, \tag{2}$$

In this case $X(t) \ge 0$, for all t.

If $\delta = n$ (a positive integer), then $X(t)$ can be written as $B_1^2 + \cdots + B_n^2$, where B_i are independent copies of Brownian Motion. In other words, $(B_1, \ldots, B_n)$ is n–dimensional Brownian Motion.

$\nu = \delta/2 - 1$ is called index of the process.

The behavior of this process depends on δ. If $\delta \ge 2$, then the process $X(t)$, will never hit zero, and if $0 < \delta < 2$, then zero, although almost surely accessible, acts as an instantaneously reflecting barrier. If $\delta = 0$, then zero is an absorbing barrier.

Let $0 \le \delta < 2$, and $x > 0$, then the first hitting time T of zero has the density:

$$f_T(t) = \frac{1}{t\Gamma(\hat{\nu})} \left(\frac{x}{2t}\right)^{\hat{\nu}} \exp\left(-\frac{x}{2t}\right), \text{ where } \hat{\nu} = \hat{\delta}/2 - 1 \text{ and } \hat{\delta} = 4 - \delta. \tag{3}$$

Calculations concerning hitting time of any other barrier are much more complicated as explained in (Göing and Yor, 1999).

We have

$$E_x^\delta(\exp -\lambda X(t)) = (1+2\lambda t)^{-\frac{\delta}{2}} \exp\left[\frac{-\lambda x}{(1+2\lambda t)}\right], \text{ and for } \delta > 0, x > 0 \tag{4}$$

the transition density at $t > 0$ is

$$q_t^\delta(x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left[\left(-\frac{(x+y)}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{t}\right)\right], y \geq 0,$$

and

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

More generally, for any $a \geq 0$, $b \neq 0$

$$\begin{aligned} & E_x^\delta \exp(-aX(t) - \frac{b^2}{2}\int_0^t X(s)ds \\ = \; & \left[\cosh(bt) + 2ab^{-1}\sinh(bt)\right]^{-\frac{\delta}{2}} \exp\left\{-\frac{xb[1+2ab^{-1}\coth(bt)]}{2\,(\coth(bt)+2ab^{-1)}}\right\}. \end{aligned} \tag{5}$$

We have two important properties:

i) Scaling. If X is $BESQ^\delta(x)$, then for any $c > 0$, the process $c^{-1}X(ct)$ is $BESQ^\delta(x/c)$.

ii) Additivity, sometimes called Pythagoras theorem

$$BESQ^\delta(x) * BESQ^\delta(x') = BESQ^{\delta+\delta'}(x+x'), \tag{6}$$

where "$*$" means independent sum.

Note: The square root of a squared Bessel process, called Bessel process plays an important role in pricing Asian and passport options.

If $\delta \geq 2$, $x > 0$, then zero is inaccessible. Therefore, by Itô formula, we have for

$$\begin{aligned} U(t) &= \sqrt{X(t)}, \\ dU(t) &= dW(s) + \frac{\delta - 1}{2}\int_0^t (U(s))^{-1} ds. \end{aligned} \tag{7}$$

The next class of processes will be squared Bessel processes with linear drifts.

We define $BESQ^{\delta}_{\beta}(x)$, $x \geq 0$ as the unique strong solution of

$$dY(t) = 2\sqrt{Y(t)}dW(t) + (2\beta Y(t) + \delta)dt, Y(0) = x \tag{8}$$

The family of this processes also satisfies the Pythagoras theorem but with the same β in both cases:

$$BESQ^{\delta}_{\beta}(x) * BESQ^{\delta'}_{\beta}(x') = BESQ^{\delta+\delta'}_{\beta}(x + x'). \tag{9}$$

There are two basic techniques to obtain the process $Y(t)$ from $X(t)$, which is $BESQ^{\delta}(x)$.

i) Change of time

$$Y(t) = e^{2\beta t} X\left(\frac{1 - e^{-2\beta t}}{2\beta}\right). \tag{10}$$

ii) Change of law by Girsanov's theorem.

Denote by ${}^{\beta}Q^{\delta}_{x}$ (Q^{δ}_{x} resp.) the law of the processes $BESQ^{\delta}_{\beta}(x)$ and ($BESQ^{\delta}(x)$ resp.)

Let $\mathcal{F}_t = \sigma(X(s), 0 \leq s \leq t)$.

Then on $\mathcal{F}_t$

$$\frac{d^{\beta}Q^{\delta}_{x}}{dQ^{\delta}_{x}} = \exp\left(\frac{\beta}{2}[X(t) - x - \delta t] - \frac{\beta^2}{2}\int_0^t X(s)ds\right), \tag{11}$$

which is a martingale.

From i) we obtain the transitions density for the process $Y(t)$, where $\beta \neq 0$, $\delta > 0$, $x > 0$.

$${}^{\beta}q^{\delta}_{t}(x, y) = \frac{\beta}{2\sinh(\beta t)}\left(\frac{y}{x}\right)^{\nu/2} \exp\left[-\beta(1+\nu)t + \frac{xe^{\beta t} + ye^{-\beta t}}{2\sinh(\beta t)}\right] I_{\nu}\left(\frac{\beta\sqrt{xy}}{\sinh(\beta t)}\right) \tag{12}$$

for any $y \geq 0$

If $\beta < 0$ we obtain the Cox, Ingersoll & Ross model (*CIR*) (with $\sigma = 2$ and different then usual parametrization).

For $\delta < 2$, $-\beta > 0$ we have the density of the first hitting time T of 0:

$${}^{-\beta}\rho^{\delta}_{T}(t) = \frac{x^{1-\delta/2}}{2^{\nu}\Gamma(\nu)} \exp\left[\frac{\beta}{2}(\delta t + x(1 - \coth(\beta t)))\right]\left[\frac{\beta}{\sinh(\beta t)}\right]^{\frac{4-\delta}{2}}, \tag{13}$$

$x > 0$, and $\nu = \frac{4-\delta}{2} - 1$.

There are many further extensions of the $BESQ^{\delta}$ processes with more or less explicit calculations.

Consider the process $Y(t)$:

$$dY(t) = 2\sqrt{Y(t)}dW(t) + (2\beta_t Y(t) + \delta)dt, \tag{14}$$

and $Z(t) = \sigma_t Y(t)$, where $\beta_t \geq 0$, and $\sigma_t > 0$ are deterministic differenciable functions.

Then

$$dZ(t) = 2\sqrt{\sigma_t Z(t)}dW(t) + \left\{ \left[2\beta_t + \frac{\sigma_t'}{\sigma_t} \right] Z(t) + \delta\sigma_t \right\} dt. \tag{15}$$

Now we have:

$$\begin{aligned} E(\exp(-\int_0^u Z(s)ds) &= \exp\left\{ \left[\ln\varphi_u(u) - \int_0^u \beta_s ds \right] \delta + x(-\beta_0 + \varphi_u'(0)) \right\}, \\ x &= \frac{Z(0)}{\sigma_0}. \end{aligned} \tag{16}$$

Where in $(0, u)$, $\varphi_u(s) = \varphi(s)$ satisfies:

$$\begin{aligned} \frac{\varphi''(s)}{\varphi(s)} &= \beta_s^2 + \beta_x' + 2\sigma_s, \\ \frac{\varphi^{-}(u)}{\varphi(u)} &= \beta_u, \quad \varphi(0) = 1, \quad \text{and} \end{aligned}$$

φ^{-} is the left hand side derivative. Calculations can be found in (Szatzschneider, 2002), and in general in (Revuz and Yor, 1998).

Another extension of the process $Y(t)$ considers a stochastic (adapted) process δ_s, instead of δ.

$$dZ(s) = 2\sqrt{Z(s)}dW(s) + (\delta_s + 2\beta Z(s))ds, \quad \beta < 0, \quad \delta_x > 0, \qquad \text{and} \tag{17}$$

$\delta_s > 0$ is called the stochastic reversion–level–determining process:

As before we will denote its law ${}^{\beta}Q_x^{\delta}$.

We will write here the law of

$$Z(t) = e^{2\beta t} X \left(\frac{1 - e^{-2\beta t}}{2\beta} \right) \tag{18}$$

(being $X(t)$, $BESQ^{\delta}(x)$), which is ${}^{\beta}Q_x^{\delta'}$ with

$$\delta'(u) = \delta\left(\frac{1-e^{-2\beta u}}{2\beta}\right). \tag{19}$$

If

$$dX(t) = 2\sqrt{X(t)}dW(t) + \delta(t)dt \tag{20}$$

where $\delta(t)$ is a deterministic function of time, $X(0) = x > 0$, then

$$E_x(\exp -\lambda X(t)) = \exp\left\{-\lambda\frac{x}{1+2\lambda t} - \int_0^t \frac{\lambda\delta(s)}{1+2\lambda(t-s)}ds\right\}. \tag{21}$$

The process called double square root model, is defined in (Longstaff, 1989):

$$\begin{aligned} dr(t) &= 2\sigma\sqrt{r(t)}dW(t) + (1-\kappa\sqrt{r(t)} + 2\beta r(t))dt \\ & \quad \kappa > 0, \beta < 0. \end{aligned} \tag{22}$$

It seems impossible to find explicitly

$$E\left(\exp - \int_0^t r(s)ds\right)$$

in this model.

If t is replaced by τ, then calculations are semi explicit (in the sense that involves very complicated integrals). Here τ is an exponential variable independent of the process.

Finally we will mention squared Bessel processes with negative dimensions:

$$dX(t) = 2\sqrt{|X(t)|}dW(t) + \delta dt, \quad \text{where } \delta < 0, X(0) = x > 0. \tag{23}$$

In this case the Pythagoras theorem is no longer valid, an the analysis of these processes becomes more difficult, cf (Göing and Yor, 1999).

Wojciech Satzschneider

References

Cox, J., J. Ingersoll, S. Ross, “A theory of the term structure of interest rates”, Econometrica, **53(2) (1985)**:385[–]407.

Göing, A., M. Yor, “A survey and some generalizations of Bessel processes”, http://www.risklab.ch/papers.htm

Longstaff, Francis, “A nonlinear general equilibrium model of the term structure of interest rates”, *Journal of Financial Economics*, **23(1989)**: 195[–]224.

Revuz, D., M. Yor, *“Continuos Martingales and Brownian Motion”*, Berlin, Heidelberg, and New York, Springer, **1998**.

Szatzschneider, Wojciech, “Extended Cox, Ingersoll & Ross Model” *Revista Mexicana de Economía y Finanzas*, **1(4)(2002)**.

See also: COX–INGERSOLL–ROSS (CIR) INTEREST RATE MODEL (1995); HULL AND WHITE MODEL; STOCHASTIC PROCESSES IN FINANCIAL MARKETS