

Extended Cox, Ingersoll & Ross Model

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Abstract. The study is dedicated to the easiest construction of the extended Cox, Ingersoll & Ross model for term structure of interest rate, and to the easiest way of pricing general interest rate derivatives within this model. To price bonds, we calculate the Laplace transform of functionals of the “elementary process”, which is squared Bessel process.

This approach, which strongly takes advantage of martingales and the Girsanov’s theorem, allows for the presentation of results in a “user friendly” form. Furthermore, we propose and apply an elementary method for the calibration of the one factor ECIR model using the history of the bonds prices.

Key words: Term structure of interest rates, Bessel processes, Girsanov theorem, time transformation, bonds and options pricing

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1 Introduction

In this section we offer a short presentation of both the Gaussian and squared Gaussian models for term structure of interest rates.

These models correspond to the first and second orders of complexity of random variables measurable with respect to $\sigma\{W(s), s \geq 0\}$, see Yor (1994), and might be seen as the simplest models for a short rate.

Following Rogers (1995), the extended Vasicek (EV) model in the risk-neutral world is defined by

$$dr(t) = \tilde{\sigma}_t dW(t) + \left(\tilde{\alpha}_t - \tilde{\beta}_t r(t) \right) dt, \quad (1)$$

and the extended Cox, Ingersoll & Ross model (ECIR) by

$$dr(t) = \tilde{\sigma}_t \sqrt{r(t)} dW(t) + \left(\tilde{\alpha}_t - \tilde{\beta}_t r(t) \right) dt. \quad (2)$$

Here $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\sigma}$ are positive and locally bounded (bounded in any finite interval) functions.

Furtheron we will use a different parametrization of ECIR surfacing naturally from our approach.

These models were considered by Hull & White (1990) who generalized models constructed by Vasicek (1977), and Cox, Ingersoll & Ross (1985).

It is easy to see, that both models are affine in the short rate in the sense that $P(t, T) = A(t, T)e^{-r_t B(t, T)}$, where $P(t, T)$ is the price at time t of the bond maturing at T .

The condition

$$\frac{\tilde{\alpha}_t}{\tilde{\sigma}_t^2} = \text{constant} \quad (3)$$

called constant dimension condition, was considered in the case of ECIR by Jamshidian, cf. Rogers (1995). Precisely this same condition will appear in our approach although it can be extended to the general case.

Even in the original CIR model with constant parameters many different generalized χ^2 distributions that appears in various papers and textbooks (even pricing simple options) may produce an aversion to put this model into practice.

Our goal is to simplify things and show that pricing problems in ECIR are relatively easy. Even the statistical procedures to determine underlying parameters (functions of time) can be performed smoothly.

In the following subsection, we will provide the description of our approach.

1.1 Description of a basic model

We define, the basic process X_t as a δ - dimensional squared Bessel process-BESQ $^\delta$, that is the unique strong solution of the stochastic differential equation.

$$dX_t = 2\sqrt{X_t} dW(t) + \delta dt. \quad (4)$$

Let us recall the Shiga & Watanabe theorem, sometimes called Pythagoras theorem: $X_t^{\delta_1} \oplus X_t^{\delta_2} \stackrel{D}{=} X_t^{\delta_1 + \delta_2}$, \oplus standing for independent sum. This theorem will play an important role in our modeling and pricing.

1.2 ECIR Model redefined

We will show in section 1.3 that in taking $X_t = BESQ^\delta$ (δ - dimensional squared Bessel process) as a basic model, and applying a change of measure, we will have the process Y_t that satisfies

$$dY_t = 2\sqrt{Y_t} dW(t) + (2\beta_t Y_t + \delta) dt, \quad Y_0 = X_0. \quad (5)$$

Here $\beta(t) < 0$ is a locally bounded and a C^1 function.

We will later drop the latter condition, and even allow for discontinuities. In this moment we prefer a model with this restriction in order to make our arguments easier.

Moreover, we multiply the process Y_t by a deterministic function $\sigma_t > 0$, σ_t locally bounded.

If the function $\sigma_t \in C^1$ (we do not require this condition), then $r(t) = \sigma_t Y_t$ satisfies

$$dr(t) = 2\sqrt{\sigma_t r(t)} dW(t) + \left\{ \left[2\beta_t + \frac{\sigma'_t}{\sigma_t} \right] r(t) + \delta\sigma_t \right\} dt, \quad (6)$$

and condition (3) is satisfied.

Now there exist unique functions β , σ and a constant δ that match the future market specifications; that is, the prices of bonds and their volatilities. For details we once more refer to Rogers, who takes the original CIR as a base and “perturbs” the process through

- deterministic C^2 time change
- multiplication by a deterministic C^1 function.

The class of processes obtained by his method is the same as in our approach. The time-change technique is also used by Maghsoody (1996).

We will use the time change but only to find the corresponding probability laws needed in option pricing, and not to construct the model.

1.3 Change of measure

Let X_t be BESQ $^\delta$ process.

Consider the continuous exponential local martingale

$$\bar{Z}_t = \mathcal{E} \left(\int_0^\bullet \beta_s \sqrt{X_s} dW_s \right)_t = \exp \left\{ \frac{1}{2} \int_0^t \beta_s dM_s - \frac{1}{2} \int_0^t \beta_s^2 X_s ds \right\}, \quad (7)$$

where $M(s) = X_s - \delta s$.

It results immediately that

$$\bar{Z}_t = \exp \left\{ \frac{1}{2} \left[\beta_t X_t - \beta_0 X_0 - \delta \int_0^t \beta_s ds - \int_0^t (\beta_s^2 + \beta'_s) X_s ds \right] \right\}. \quad (8)$$

If $\beta_s^2 + \beta'_s \geq 0$, the local martingale \bar{Z}_t is bounded ($X_s \geq 0$ and $\beta_t < 0$), and therefore is martingale. If not, we can use a slightly more elaborated argument, called Novikov criterion for small intervals.

Therefore in any case \bar{Z}_t is a martingale, and the change of drift via the Girsanov’s theorem is justified.

2 Pricing Bonds in ECIR model

We will now show how to price zero-coupon bonds in the ECIR model, at time zero (to simplify the notation).

We have

$$\begin{aligned}
P(0, u) &= E \left[\exp - \int_0^u r(s) ds \right] \\
&= E \left[\exp - \int_0^u Y_s \sigma_s ds \right] \\
&= E \left[\exp \left(- \int_0^u X_s \sigma_s ds \right) \cdot \bar{Z}(u) \right] \\
&= \exp \left[\frac{1}{2} (-\beta_0 X_0 - \delta \int_0^u \beta_s ds) \right] \\
&\quad \cdot E \left\{ e^{\frac{1}{2} [\beta_u X_u - \int_0^u X_s (\beta_s^2 + \beta_s' + 2\sigma_s) ds]} \right\}. \tag{9}
\end{aligned}$$

We look for $F_u(s) = F(s)$ such that $F^2(s) + F'(s) = \beta_s^2 + \beta_s' + 2\sigma_s$, for $s \in [0, u)$. $F(u) = \beta_u$. Such F exists and is of finite variation.

Moreover it is easy to solve this equation by numerical methods.

Now we can apply the approach from section 1.3 with F playing the rôle of β , and the expectation in (9), completed with the corresponding deterministic function

$$\exp \left\{ \frac{1}{2} \left(-F(0)X_0 - \delta \int_0^t F(s) ds \right) \right\}$$

would be 1.

In this part the most suitable approach to the Riccati equation, that defines F , is via the corresponding Sturm-Liouville equation.

Writing $F(s) = \frac{\varphi'(s)}{\varphi(s)}$, $s \in [0, u)$, $\varphi(0) = 1$ we have

$$\frac{\varphi''(s)}{\varphi(s)} = h(s) = \beta_s^2 + \beta_s' + 2\sigma_s \text{ in } [0, u), \frac{\varphi^-(u)}{\varphi(u)} = \beta_u,$$

where φ^- is the left-hand side derivative.

It is easy to see that

$$\frac{\varphi'(s)}{\varphi(s)} < \beta_s \text{ for } s \in [0, u). \tag{10}$$

The exponential martingale that corresponds to F will be called Z

$$Z_t = \mathcal{E} \left(\int_0^t F(s) \sqrt{X(s)} dW_s \right)_t \tag{11}$$

(Z corresponds to F in the same way as \bar{Z} to β). Whenever necessary we will stress the dependence of Z on u and write Z_u

Note that the equation $\frac{\varphi''}{\varphi} = c^2$ (we consider a positive constant) can be solved easily and we will do it in a while; so if $\beta_s^2 + \beta_s' + 2\sigma_s$ is constant, then the prices of bonds can be found explicitly.

The solution of the equation $\frac{\varphi''(s)}{\varphi(s)} = c^2$ in $[0, u]$ is clearly $A e^{cs} + B e^{-cs}$ with conditions $A + B = 1$ (because of $\varphi(0) = 1$) and $c \left(\frac{A e^{cu} - B e^{-cu}}{A e^{cu} + B e^{-cu}} \right) = \beta_u$.

Finally, in the general case, we obtain after elementary calculations

$$P(0, u) = \exp \frac{1}{2} \left\{ \left[\ln \varphi_u(u) - \int_0^u \beta_s ds \right] \delta + x (-\beta_0 + \varphi'_u(0)) \right\}, \quad (12)$$

$$x = \frac{r_0}{\sigma_0}.$$

We set here φ_u to stress the dependence on u of the solution $\varphi(s)$ in $[0, u]$. In the same way, $F = F_u$.

Using the comparison criterion for diffusion processes, or directly from (10), we obtain the intuitively obvious fact that the prices of bonds are decreasing (deterministic) functions of the short rate, and the explicit form for A and B in the affine representation:

$$P(0, u) = A(0, u) e^{-B(0, u)r(0)} \quad (13)$$

Our main result may be written easily as:

$$E_x^\delta \left[\exp \left(-\frac{1}{2} \int_0^u X_s \mu(ds) \right) \right] = \exp \left\{ \frac{1}{2} [\varphi'_\mu(0)x + \delta \ln \varphi_\mu(u)] \right\} \quad (14)$$

where μ is the measure with density $h(s)$ in $[0, u]$, and the atom $\beta(u)$ at u , $x = X_0 = \frac{r(0)}{\sigma_0}$, $\varphi = \varphi_\mu$. In the case of $h > 0$ this is precisely Cameron-Martin-Pitman-Yor's formula!, cf. Revuz & Yor (1998).

Note: It is well known that to price bonds in the ECIR model one has to solve a Riccati equation, cf. Rogers (1995), but the approach presented here is via the corresponding Sturm-Liouville equation.

Now we will show how easy our approach results in the case of piecewise-constant parameters β_s and σ_s (this because of our parametrization of ECIR).

This case was considered by Schlögl & Schlögl (1997) using a different approach involving many different χ^2 distributions. We will show how to price bonds $P(0, t)$ in the case of $\beta_s = \beta_i$, and $\sigma_s = \sigma_i$ for $s \in [t_i, t_{i+1})$, $i = 0, 1, \dots, n-1$, $t_n = t$, $\beta_t = \beta_n$. As before $\beta_i < 0$, $\sigma_i > 0$.

Now

$$\int_0^t X(s) d\beta_s = \sum_{i=1}^n (\beta_i - \beta_{i-1}) X_{t_i}$$

and

$$P(0, t) = E \left[\exp \frac{1}{2} \{ \beta_t X_t - \beta_0 X_0 - \delta \int_0^t \beta_s ds - \sum_{i=1}^n \left[(\beta_{i-1}^2 + 2\sigma_{i-1}) \int_{t_{i-1}}^{t_i} X(s) ds \right] - \sum_{i=1}^n (\beta_i - \beta_{i-1}) X_{t_i} \} \right]. \quad (15)$$

As is expected, $\beta_n = \beta_t$ are canceled (changing β in one ending point does not affect the price of a bond). Note that we have chosen the right continuous version of β . It is clear that we solve the problem of bond pricing if we find the function $\varphi(s)$, $s \in [0, t]$, such that:

- i) $\varphi(s)$ is continuous,
- ii) $\frac{\varphi''(s)}{\varphi(s)} = \beta_{i-1}^2 + 2\sigma_{i-1}$ in (t_{i-1}, t_i)
- iii) $\frac{\varphi^+(t_i) - \varphi^-(t_i)}{\varphi(t_i)} = \beta_i - \beta_{i-1}$
- iv) $\frac{\varphi^-(t_n)}{\varphi(t_n)} = \beta_{n-1}$
- v) $\varphi(0) = 1$

Here $i = 1, 2, \dots, n-1$, and φ^+ (φ^-) stands for right (left) hand side derivative respectively.

Because the general solution of $\frac{\varphi''(s)}{\varphi(s)} = c_{i-1}^2$ is $A_{i-1}e^{c_{i-1}s} + B_{i-1}e^{-c_{i-1}s}$, the problem is reduced to the solution of $2n$ linear equations for A_{i-1}, B_{i-1} , $i = 1, \dots, n$. Recalling our general formula for $P(0, t)$ we only need to find A_0 and A_{n-1} because $B_0 = 1 - A_0$, and B_{n-1} is recovered from the condition iv).

$$\frac{\varphi^-(t_n)}{\varphi(t_n)} = \beta_{n-1}.$$

Important remark

To calibrate the model it is more convenient to consider a function right continuous $\tilde{\varphi}$ such that $\tilde{\varphi}(t_{i-1}) = 1$, for $i = 1, \dots, n$. In this case $\tilde{\varphi}$ will not be continuous and instead of $\tilde{\varphi}_u(u)$ in the pricing formula (12) we will have

$$\prod_{i=1}^n \tilde{\varphi}_{t_i}(t_i^-).$$

where $\tilde{\varphi}(t^-)$ stands for the left side limit. The pricing formula becomes less elegant but more efficient because, in this case, there is no need to solve the system of equations. We will be back to this approach in section 5.

3 Options on zero-coupon bonds

The price of an option over a bond— $h(P(t, u))$ may be written as;

$$P(0, t) \cdot \frac{E\left(e^{-\int_0^t r(s)ds} h(P(t, u))\right)}{P(0, t)} = P(0, t) \cdot E_t^*(h(P(t, u))).$$

This is because $\frac{e^{-\int_0^t r(s)ds}}{P(0, t)}$ is a density function.

The corresponding P_t^* probability is called a t -forward martingale measure. Later on, we will find the t -forward martingale measure for the ECIR model.

In the sequell we will find the transition density of the process $U(t)$ defined by

$$U(t) = x + 2 \int_0^t \sqrt{U(s)} dB(s) + 2 \int_0^t \left(F_u(s)U(s) + \frac{1}{2}\delta \right) ds$$

The law of the process $U(t)$, $t \leq u$ is the law of X_t under the change of measure $Q_x^\delta \rightsquigarrow Z_u Q_x^\delta$ where Q_x^δ is the law of $BESQ^\delta$ and Z is defined by formula (10).

We will find the law of U using the deterministic time change, and write $F = F_u$.

First set $\delta = 1$. If $V(t) = v + W(t) + \int_0^t F(s)V(s)ds$ then $U = V^2$ is the solution of

$$U(t) = x + 2 \int_0^t \sqrt{U(s)} \operatorname{sgn} V(s) dW(s) + 2 \int_0^t F(s)U(s)ds + t, v = \sqrt{x},$$

and $\int_0^t \operatorname{sgn} V(s) dW_s$ is another Brownian Motion.

But the equation

$$V(t) = v + W_t + \int_0^t F(s)V(s)ds$$

is a well known linear equation and its solution is

$$V(t) = \varphi(t) \left[v + \int_0^t \varphi^{-1}(s) dW_s \right], \quad \frac{\varphi'(s)}{\varphi(s)} = F(s), \quad \varphi(0) = 1. \quad (16)$$

Because the stochastic integral of a deterministic function is Gaussian, we obtain that $V(t)$ is a Gaussian process with, for a given t , mean $v\varphi(t)$ and variance ζ^2 , where

$$\zeta^2 = \zeta^2(t) = \varphi^2(t) \int_0^t \varphi^{-2}(s) ds.$$

Hence, still using $\delta = 1$, the transition density of the process $U(t)$ is

$$r_t^{1,\varphi}(x, \cdot) = q_{\zeta^2}^1(x\varphi^2(t), \cdot),$$

where q^1 is the transition density of W_t^2 (which is the squared Bessel process of dimension 1).

Therefore $r_t^{\delta,\varphi}(x, \cdot) = q_{\zeta^2}^\delta(x\varphi^2(t), \cdot)$, where q^δ is the transition density of the squared δ -dimensional Bessel process. This fact results easily from the Pythagoras theorem (section 1.1) applied to the law of the process U . The proof is exactly the same as for BESQ processes.

Finally we write well known formulas:

$$q_t^\delta(x, y) = \frac{1}{2t} \left(\frac{y}{x} \right)^{\frac{\nu}{2}} \exp\left(-\frac{(x+y)}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{t}\right),$$

$t > 0$, $x > 0$, and I_ν is the Bessel function of index $\nu = \frac{\delta}{2} - 1$

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n! \Gamma(\nu + n + 1)}.$$

After these preliminaries, we will solve the problem of option pricing.

Applying the same method as in the preceding section we have (see formula 12)

$$P(\theta, T) = e^{\frac{1}{2}\delta [\ln \tilde{\varphi}(T) - \int_\theta^T \beta_s ds]} \cdot e^{\frac{1}{2} \cdot \frac{r_\theta}{\sigma_\theta} (-\beta_\theta + \tilde{\varphi}'(\theta))} \quad (17)$$

where in (θ, T)

$$\frac{\tilde{\varphi}''(s)}{\tilde{\varphi}(s)} = h(s), \quad \tilde{\varphi}(\theta) = 1, \quad \frac{\tilde{\varphi}'(T)}{\tilde{\varphi}(T)} = \beta_T$$

Therefore in $[\theta, T]$, $\tilde{\varphi}(s) = \frac{\varphi_T(s)}{\varphi_T(\theta)}$.

Now $(P(\theta, T) - k)_+ = (P(\theta, T) - k) \cdot I[Y_\theta < k']$ where

$$k' = \frac{2 \left\{ \ln k + \frac{1}{2}\delta \left[\int_\theta^T \beta(s) ds - \ln \varphi_T(T) + \ln \varphi_T(\theta) \right] \right\}}{\frac{\varphi_T'(\theta)}{\varphi_T(\theta)} - \beta(\theta)}, \quad (18)$$

$k' > 0$, $Y_\theta = \frac{r(\theta)}{\sigma_\theta}$.

In the following propositions, for $u = \theta$ we will write ψ , and for $u = T$ we will write φ , instead of φ_θ and φ_T respectively.

Define two probabilities P_1 , P_2 with densities with respect to the original probability law P of the process Y_s , $s \leq \theta$ and $s \leq T$ respectively, as follows;

$$\begin{aligned}\frac{dP_1}{dP} &= \frac{\exp\left(-\int_0^\theta Y_s \sigma_s ds\right)}{P(0, \theta)} \\ \frac{dP_2}{dP} &= \frac{\exp\left(-\int_0^\theta Y_s \sigma_s ds\right) P(\theta, T)}{P(0, T)}.\end{aligned}$$

Now the following pricing formula holds

Proposition 3.1 The price C splits into two terms

$$C = P(0, T) \cdot P_2\left(\frac{r(\theta)}{\sigma_\theta} < k'\right) - kP(0, \theta) \cdot P_1\left(\frac{r(\theta)}{\sigma_\theta} < k'\right) \quad (19)$$

where under the law P_1 , $\frac{r(\theta)}{\sigma_\theta}$ has the density $r^\delta(x, \cdot) = q_{\zeta^2}^\delta(x\psi^2(\theta), \cdot)$ with $x = \frac{r(0)}{\sigma_0}$, and $\zeta^2 = \psi^2(\theta) \cdot \int_0^\theta (\psi(s))^{-2} ds$, while in the analogous formula for the law P_2 we replace ψ by φ .

Proof.

$$\begin{aligned}P_1\left(\frac{r(\theta)}{\sigma_\theta} < k'\right) &= P_1(Y_\theta < k') \\ &= E\left[\frac{\exp\left\{-\int_0^\theta X_s \sigma_s ds \cdot \bar{Z}_\theta \cdot I[X_\theta < k']\right\}}{P(0, \theta)}\right] \\ P_2\left(\frac{r(\theta)}{\sigma_\theta} < k'\right) &= P_2(Y_\theta < k') \\ &= E\left[\frac{\exp\left\{-\int_0^\theta X_s \sigma_s ds \cdot P(\theta, T) \cdot \bar{Z}_\theta \cdot I[X_\theta < k']\right\}}{P(0, T)}\right].\end{aligned}$$

(In the formula for $P(\theta, T)$ we clearly change $Y(\theta) = \frac{r(\theta)}{\sigma(\theta)}$ for $X(\theta)$).

Trivial algebra leads to

$$P_i(Y_\theta < k') = E[Z_i \cdot I(X_\theta < k')],$$

where Z_i is an exponential martingale.

$$Z_i = \exp\left\{\frac{1}{2}\left[-\delta \ln \gamma_i(\theta) + X_\theta \frac{\gamma_i'(\theta)}{\gamma_i(\theta)} - \gamma_i'(0)x - \int_0^\theta X_s h(s) ds\right]\right\},$$

$\gamma_1 = \psi$ and $\gamma_2 = \varphi$. Therefore $Z_1 = Z_\theta$ in our previous “official” notation.

Generalizing we have the following proposition:

Proposition 3.2

$$E \left(H(r(\theta)) \cdot \exp \left(- \int_0^\theta r(s) ds \right) \right) = P(0, \theta) \cdot E_\theta^* (H(r(\theta))), \quad (20)$$

where E_θ^* is the expectation with respect to the probability distribution with density

$$q_{\zeta^2}^\delta (x\psi^2(\theta), \cdot).$$

Proof.

$$\begin{aligned} E \left(H(Y_\theta \sigma_\theta) \cdot e^{-\int_0^\theta Y_s \sigma_s ds} \right) &= E \left\{ H(X_\theta \sigma_\theta) \cdot \bar{Z}_\theta e^{-\int_0^\theta X_s \sigma_s ds} \right\} \\ &= P(0, \theta) \cdot E \{ H(X_\theta \sigma_\theta) \cdot Z_1 \} \\ &= P(0, \theta) \cdot E_\theta^* (H(r(\theta))) \end{aligned}$$

E_θ^* is precisely the expectation with respect to the θ -forward martingale measure. This forward martingale measure was obtained here in a straightforward way. Compare with Maghsoodi (1996).

4 Multifactor models

The one-factor model is often criticized because of the fact that prices of bonds of different maturities are perfectly correlated. Prices are deterministic functions of $r(t)$ (if we value bonds at time t).

Pricing interest rate derivatives with multifactor models in the case of independent factors does not present additional analytical difficulties, but clearly numerical calculations will be more complicated.

We will consider multi= 2, and begin with independent factors. Let $r(t) = r_1(t) \oplus r_2(t)$. As in the section 1 “ \oplus ” means the sum of independent terms. The process $r(t)$ becomes a one factor model if and only if

$$\sigma_1(t) = \sigma_2(t), \quad \beta_1(t) = \beta_2(t).$$

Otherwise, the general valuation formula has the form

$$\begin{aligned} C &= E \left[e^{-\int_0^\theta r(s) ds} H(r_1(\theta), r_2(\theta)) \right] \\ &= P(0, \theta) \cdot E_\theta^* [H(r_1(\theta), r_2(\theta))], \end{aligned}$$

where E_θ^* is the expectation with respect to the modified independent process. Through the usual conditioning technique the pricing problem might be reduced to univariate integrations.

The special structure of the CIR model, and particularly the Shiga & Watanabe theorem, allows a relatively easy approach even in the case of dependent factors.

The general multifactor ECIR model is defined in Rogers (1995). We propose here an alternative approach which allows for dependency of factors measured by corresponding correlations.

We will show here the general procedure of reducing the problem in our setting to independent factors with the sole application to pricing zero-coupon bonds. The term “factor” will be slightly more general than before.

Let

$$Y_i(t) = 2\sqrt{Y_i(t)} dW_i(t) + (2\beta_i(t)Y_i(t) + \delta_i) dt.$$

An easy application of the Shiga & Watanabe theorem shows that $Y_i(t)$ might be decomposed into the sum of independent processes $Y_i^1(t)$, $Y_i^2(t)$ with underlying Brownian Motion $W_i^{(1)}$ independent of $W_i^{(2)}$.

We assume that $W_1^{(2)}$ and $W_2^{(2)}$ are independent, and the dependency between $r_1(t)$ and $r_2(t)$ will result from the dependency between Y_1^1 and Y_2^1 , both driven by $W = W_1^{(1)} = W_2^{(1)}$.

The bond price $P(0, T)$ becomes

$$E \left[e^{-\int_0^T (\sigma_1(s)Y_1^1(s) + \sigma_2(s)Y_2^1(s)) ds} \right] \cdot E \left[e^{-\int_0^T \sigma_1(s)Y_1^2(s) ds} \right] \cdot E \left[e^{-\int_0^T \sigma_2(s)Y_2^2(s) ds} \right] \quad (21)$$

The analytical method for the calculation of the first term is based on the space-time transformation of a squared Bessel process. The method might be performed in a more general setting, but our presentation (for simplicity) will only consider r_1^1 and r_2^1 , with constant parameters $\beta_1, \beta_2 < 0$.

We assume $\delta_1^{(1)} = \delta_2^{(1)} = \delta$. In this case we set $Y_i^1(t) = e^{2\beta_i t} X \left(\frac{1 - e^{-\beta_i t}}{2\beta_i} \right)$, where X is the δ -dimensional squared Bessel process (see the beginning of section 3).

Now

$$\int_0^t \sigma_1(s) Y_1^1(s) ds = \int_0^t \sigma_1(s) e^{2\beta_1 s} X \left(\frac{1 - e^{-2\beta_1 s}}{2\beta_1} \right) ds.$$

The deterministic function

$$C(t) = \frac{1 - e^{-2\beta_1 t}}{2\beta_1}$$

is a very simple time change, and $C(0) = 0$.

Now

$$\int_0^t \sigma_1(s) Y_1^1(s) ds = \int_0^{C(t)} (1 - 2\beta_1 u)^{-2} \sigma_1 \left[\frac{\ln(1 - 2\beta_1 u)}{-2\beta_1} \right] X(u) du.$$

Therefore the first term in (21) might be calculated as in section 1.

Assuming $\beta_2 < \beta_1 < 0$, after straightforward calculations we obtain

$$\text{Cov}(r_1(t), r_2(t)) = 2\delta\sigma_1(t)\sigma_2(t)e^{2(\beta_1+\beta_2)t} \left(\frac{1 - e^{-2\beta_1 t}}{2\beta_1} \right)^2. \quad (22)$$

This approach, if ever put into practice, requires the identification of factors as some observable variables.

We simply consider this possibility as it avoids solutions of multidimensional Riccati equations cf. Rogers (1995).

5 Calibration of the model

We remind from the section 2 the price of the Bond $P(0, t)$ is in our model

$$P(0, t) = \exp \frac{1}{2} \left\{ \delta \left[\ln \prod_{i=1}^n \tilde{\varphi}(t_i^-) - \int_0^t \beta_s ds \right] + x [-\beta_0 + \tilde{\varphi}(0)] \right\}, \quad (23)$$

$$x = \frac{r_0}{\sigma_0}.$$

5.1 The method

We choose parameter δ fixed and allow changes of β and σ every week. More precisely: At any time t , fixed day of a week, we observe prices of bonds at different maturities:

$$O(i, i + j), \text{ for } i = 1, \dots, n, \text{ and } j = 1, \dots, d.$$

For any choice of parameters we recover $r_i^{(j)}$. Usually $r_i^{(j)} \neq r_i^{(k)}$ for $k \neq j$.

This occurs because we consider one dimensional (one factor) model of d -dimensional problem.

We set $r_i = \frac{1}{d} \sum_{j=1}^d r_i^{(j)}$ as a *theoretical* value of the process at time t_i . With this r_i (and the same parameters) we choose $P(i, i + j)$ as a *theoretical* value of the corresponding bond. Our goal is to choose the ‘‘trajectory’’ of parameters that minimizes:

$$\sum_{i=1}^n \sum_{j=1}^d [P(i, i + j) - O(i, i + j)]^2.$$

To solve the problem, we use a fundamental gradient descent method, known as *Steepest Descent*. The basic idea is to move along the negative direction of the objective function.

5.2 Application

To show how this method works in practice, we consider the prices of the Mexican treasury certificates (CETES) and the prices of the American treasury bills (TBILLS) for 91, 182 and 365 days (or 13, 26 and 52 weeks) for the 130 weeks from September 11th, 1997 to March 2nd, 2000.

As an initial point to the problem, we take the CIR model parameters, obtained by a traditional method, for each of the financial instruments (CETES and TBILLS).

5.2.1 CETES

Initial (CIR) parameters:

$$\begin{aligned}\delta &= 2.24228790 \\ \beta &= -0.91145494 \\ \sigma &= 0.35791831\end{aligned}$$

We got the following results:

- The objective function value is reduced **3.7** times. From 0.00443073 (CIR) to 0.00120477 (ECIR).

The CETES bond prices adjustment performance is showed at table 5.1.

5.2.2 TBILLS

Initial (CIR) parameters:

$$\begin{aligned}\delta &= 1.42056719 \\ \beta &= -0.53749523 \\ \sigma &= 0.11286799\end{aligned}$$

After 10 iterations, we got the following result:

- The objective function value is reduced **20.2** times. From 0.00084535 (CIR) to 0.00004186 (ECIR).

The TBILLS bond prices adjustment performance is showed at table 5.2.

6 Final Remarks

We have proposed an alternative approach to the extended CIR Model, taking a squared Bessel process as the basic one.

We have shown that valuation and calibration are easier and more transparent when using our parametrization. The past and present history of prices affect future parameters (functions of time), but more studies are needed to see if our historical approach can help for further forecasting.

Our method of calibrations can be extended to other interest rate derivatives and to multifactor models, even in the case of dependent factors as in our method from section 4, based on Pythagoras' Theorem.

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	CIR $\sum \varepsilon^2$	ECIR $\sum \varepsilon^2$	CIR $\sum \varepsilon^2$ / ECIR $\sum \varepsilon^2$
13 Weeks	0.00070426	0.00028588	2.5
26 Weeks	0.00043253	0.00040726	1.1
52 Weeks	0.00329394	0.00051163	6.4
Total	0.00443073	0.00120477	3.7

Table 5.1: CETES bond prices adjustment performance.

	CIR $\sum \varepsilon^2$	ECIR $\sum \varepsilon^2$	CIR $\sum \varepsilon^2$ / ECIR $\sum \varepsilon^2$
13 Weeks	0.00003637	0.00000463	7.9
26 Weeks	0.00003714	0.00001661	2.2
52 Weeks	0.00077185	0.00002062	37.4
Total	0.00084535	0.00004186	20.2

Table 5.2: TBILLS bond prices adjustment performance.