Cox–Ingersoll–Ross (CIR) interest rate model (1985).

CIR model for instantaneous interest rates was constructed in 1980 and published in 1985.

Several mathematical results about *CIR* were known since (Feller, 1951).

CIR model is defined as the unique strong solution of

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \quad \text{where} \quad r(0), \kappa, \theta, \sigma > 0. \tag{1}$$

This is an example of square root process.

We will analyse risk premiums in the final part, so here we assume them equal zero.

Following (Cox *et al.*, 1985) the price of a zero coupon bond is

$$\begin{aligned} E(\mathrm{e}^{-\int_t^T r(s)ds}|\mathcal{F}_t) &= P(r(t), t, T) = \\ & A(t,T)\mathrm{e}^{-B(t,T)r(t)}, \qquad \text{where} \end{aligned} \tag{2}$$

$$\begin{aligned} A(t,T) &= \left[\frac{2\gamma \exp[\frac{1}{2}(\kappa+\gamma)(T-t)]}{(\gamma+\kappa)(\exp(\gamma(T-t))-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2}, \\ B(t,T) &= \frac{2(\exp(\gamma(T-t))-1)}{(\gamma+\kappa)[\exp(\gamma(T-t))-1]+2\gamma}, \\ \gamma &= (\kappa^2+2\sigma^2)^{1/2} \end{aligned} \tag{3}$$

The first equality in (2) results because $r(t)$ is a diffusion, therefore Markov process.

The second equality shows that *CIR* belongs to the so–called affine models. For discussion of continuous affine models we refer to (Duffie, 1996). General affine models are presented in (Duffie *et al.*, 2002) and (Felipović, 2001).

For $s > t$ we have

$$E(r(s)|\mathcal{F}_t) = r(t)\exp(-\kappa(s-t)) + \theta(1-\exp(-\kappa(s-t))), \tag{4}$$

$$\begin{aligned} \mathrm{Var}\,(r(s)|\mathcal{F}_t) &= r(t)\left(\frac{\sigma^2}{\kappa}\right)[\exp(-\kappa(s-t)) - \exp(-2\kappa(s-t))] + \\ & \theta\left(\frac{\sigma^2}{2\kappa}\right)(1-\exp(-\kappa(s-t))^2, \end{aligned} \tag{5}$$

and the transition density from t to s,

$$f(x,y) = ce^{-u-v}\left(\frac{v}{u}\right)^{\nu/2} I_\nu(2(uv)^{1/2}), \tag{6}$$

where

$$\nu = \frac{2\kappa\theta}{\sigma^2} - 1,$$

$$\begin{aligned} c &= \frac{2\kappa}{\sigma^2(1-\exp(-\kappa(s-t))} \\ u &= cx\exp(-\kappa(s-t)) \\ v &= cy, \end{aligned}$$

and

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(\nu+n+1)}.$$

The prices of European call option on future bond are given in (Cox *et al.*, 1985).

More general interest rate derivatives can be found using the concept of forward martingale measure P_s^*. Set for example $t = 0$:

$$E(H(r(s))\exp\left(-\int_0^s (r(u)du\right) = P(r(0),0,s)E_s^* H(r(s)), \tag{7}$$

where

$$\frac{dP_s^*}{dP_s} = \frac{\exp\left(-\int_0^s r(u)du\right)}{P(r(0),0,s)}.$$

For details in more general case (a version of Hull & White model) cf. (Szatzschneider, 2002).

The price of the consol in *CIR* model

$$f(r) = E_r\left(\int_0^\infty \left(\exp\left(-\int_0^t r(s)\right) ds\right) dt\right), r(0) = r,$$

is analysed in (Delbaen, 1993) and (Yor, 1993).

Until now we were talking about one factor *CIR* model (driven by one Brownian motion). In this case bonds prices at different maturities are totally correlated being deterministic functions of the spot price $r(t)$, while priced at time t. As commented by (Rogers, 1995) these might be forgiven, but many empirical studies suggest that multifactor models give better adjustment to data.

There are several extensions to multifactor models. In the first one consider the mean reversion level (markets can influence this level) $\theta(s)$, a stochastic process, being $\theta(s)$ another *CIR* an example. In this part it is more convenient to use the different parametrization (also used in the entry square root process). We set

$$dX_s = (2\beta X_s + \delta_s)ds + 2\sqrt{X(s)}dW(s), \tag{8}$$

δ_s is non–negative adapted process.

Under mild conditions on the process δ_s cf. (Deelstra and Delbaen, 1995), one can prove two important convergence theorems, useful for approximations. If

$$\frac{1}{s}\int_0^s \delta_u du \xrightarrow{a.e} \overline{\delta} > 0,$$

then

i) $\frac{1}{s}\int_0^s X(u)du \xrightarrow{a.e} \frac{\overline{\delta}}{2\beta}$

ii) $\left(\sqrt{\frac{-2\beta^3}{\overline{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_n}{2\beta}\right)du\right) \xrightarrow{\mathcal{L}} B_t$, where B_t is a Brownian motion, and "$\mathcal{L}$" denotes convergence in law.

A general version of multifactor *CIR* model or more generally, of Hull and White one, is presented in (Rogers, 1995), being this a sum of squares of Gaussian processes.

The case when factors can be directly observed from the current yield curve is analysed by (Duffie and Kan, 1996), and reviewed briefly in the textbook by (Duffie, 1996).

Following Duffie and Kan, and being interest rate the sum of the factors, we consider for any fixed times $\tau_1, \ldots, \tau_n$, and for each t and i, x_{it} the yield at time t on a zero–coupon bond of maturity $\tau_i + t$.

$$x_{it} = \frac{-1}{\tau_i}\log P(x, t, t + \tau_i).$$

Another case considered by (Pearson and Sun, 1994) in their statistical studies is

$$r(t) = r_1(t) * r_2(t) + \overline{R},$$

"$*$" stands for independent sum and $\overline{R}$ is a constant so it is a kind of translated 2 factors *CIR*. They set nominal interest rates as the sum of real interest rates and inflation.

Statistical studies in (Chan *et al.,* 1992) used generalized method of moments, and (Pearson and Sun, 1994) maximum likelihood method. The conclusion from these studies can be resumed that two factors are not sufficient to obtain good fit. In multifactor models with non zero correlations between factors, these correlations must be positive as written in (Dai and Singleton, 2000). Further we read that "the data on U.S. interest rates seem to call for negative correlations among risk factors". The authors conclude that these kind of models" are not consistent with the historical behavior of U.S interest rates". However it is easy to obtain negatively correlated square root diffusions considering for example $(W(t)+a)^2$ and $(W(t)-a)^2$ for large a. We belive that more studies are needed in this direction and within a framework different than one offered by Dai and Singleton. Another possibility to obtain better fit is to consider Hull and White extension of the *CIR* model. If, for example, one sets piecewise constant parameters, there is still explicit solution for bonds prices (Szatzschneider, 2002).

The final part of this presentation is about equilibrium derivation of the *CIR* model, and risk premiums. The first topic is treated extensively in the original *CIR* paper. As pointed out by Rogers many interest rates models can be justified in a similar way as equilibrium ones.

We want to clarify the topic and explain what is written on page 398 of (Cox *et al.,* 1985) about linear risk premiums.

Following (Rogers, 1995), if there exist risk neutral probabilities equivalent to the observed (it means that discounted prices of all financial assets are, under these new probabilities, local martingales), then there is an economy which supports these prices as equilibrium ones.

Consider for example one dimensional financial market (driven by one Brownian Motion) where in the observed world the asset prices follow Geometric Brownian Motion: $dS(t) = (dW(t) + \mu dt)S(t)$, and $r(t)$ is *CIR* model.

Set $Z(t) = S(t)\exp(-\int_0^t r(s)ds)$ (discounted prices).

Now $dZ(t) = Z(t)[dW(t) + \mu - r(t)]dt$.

Risk neutral probability law $\widetilde{P}$ is such that $\forall_T$, on $\mathcal{F}_T$, P and $\widetilde{P}$ are equivalent ($P \sim \widetilde{P}$), and $dZ(t) = Z(t)dW^*(t)$, where W^* is another Brownian Motion.

But in this case such $\widetilde{P}$ does not exists!

An easy argument is based on what is called explosion until $T = 1$ of the process defined by $dX(t) = dW(t) + X^2(t)dt$, (Revuz and Yor, 1998 p. 381).

But pricing interest rate derivatives in *CIR* framework one should have *CIR* in the *RNW* (Risk Neutral World) and not care about the structure of the model in, unobserved for interest rates alone, Real World.

It can be proved that if under $P : S(t)$ is a before, and

$$dr(t) = 2\sqrt{r(t)}dW(t) + \left[\delta + 2\mu\sqrt{r(t)} - (2r(t) + 2r^{3/2}(t))\right] dt, \tag{9}$$

then there exists $\widetilde{P} \sim P$ and under $\widetilde{P}$

$$dr(t) = 2\sqrt{r(t)}dW^*(t) + (\delta - 2r(t))dt, \qquad \text{it means} \quad CIR. \tag{10}$$

To avoid such complications usually different risk premiums are proposed. Let

$$dZ(t) = Z(t)dW(t) + \lambda\sqrt{r(t)}dt. \tag{11}$$

In this case *CIR* in Real World changes into another *CIR* in Risk Neutral World.

If for example under P

$$dr(t) = 2\sqrt{r(t)}dW(t) + (\delta - 2r(t))dt, \qquad \text{then under} \qquad \widetilde{P}$$

$$\begin{aligned} dr(t) &= 2\sqrt{r(t)}d\left(W^*(t) - \lambda\int_0^t \sqrt{r(s)}ds\right) + (\delta - 2r(t))dt \\ &= 2\sqrt{r(t)}dW^*(t) + (\delta + (-2\lambda - 2)r(t))dt \end{aligned} \tag{12}$$

by Girsanov's theorem.

Pricing formulas for interest rates derivatives cannot identify risk premiums. However risk premiums, in general terms, allow for better adjustment of models to data cf. (Dai and Singleton, 2000, 2002), especially with risk premiums that changes the model into the one belonging to a larger than *CIR* family of models.

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References

Cox, J., J. Ingersoll, S. Ross, “A theory of the term structure of interest rates”, Econometrica, **53(2) (1985)**:385[–]407.

Chan, K., G. Karolyi, F. Longstaff, A. Sanders, “An empirical comparision of alternative models of the short–term interest rate”, *The Journal of Finance*, **XLVII(1992)**:1209[–]1227.

Dai, Q., K. Singleton, “Specification analysis of affine term structure models”, *The Journal of Finance*, **LV(5)(2000)**:1943[–]1978.

Dai, Q., K. Singleton, “Term structure dynamics in theory and reality”, http://www.stanford.edu/~kenneths/, May 2002.

Deelstra, G., F. Delbaen, “Long term returns in stochastic interest rate models: Convergence in law”, *Stochastic and Stochastic Reports*, **5(1995)**: 253[–]277.

Delbaen, Freddy, “Consols in the *CIR* model”, *Mathematical Finance*, **3(2)(1993)**:125–134.

Duffie, Darrell, *“Dynamic Asset Pricing Theory”*, Princeton University Press, **1996**.

Duffie, D., D. Filipović, W. Schachermayer, “Affine processes and applications in finance”, http://www.princeton.edu/~dfilipov, May 2002

Duffie, D., R. Kan, “A yield–factor model of interest rates”, *Mathematical Finance*, **6(4)(1996)**:379[–]406.

Feller, Williams, “Two singular diffusion problems”, *Annals of Mathematics*, **54(1951)**:173[–]182.

Filipvić, Damir, “A general characterization of one factor affine term structure models”, *Finance Stochastic*, **5(3)(2001)**:389[–]412.

Göing, A., M. Yor, “A survey and some generalizations of Bessel processes”, http://www.risklab.ch/papers.htm

Longstaff, Francis, “A nonlinear general equilibrium model of the term structure of interest rates”, *Journal of Financial Economics*, **23(1989)**: 195[–]224.

Pearson, N., T. Sun, “Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll, and Ross model”, *The Journal of Finance*, **XLIX(4)(1994)**:1279[–]1304.

Revuz, D., M. Yor, *“Continuos Martingales and Brownian Motion”*, Berlin, Heidelberg, and New York, Springer, **1998**.

Rogers, L.C.G., “Which model for term–structure of interest rates should one use?” in *Mathematical Finance*, edited by Davis, Duffie, Fleming and Shreve, Springer–Verlag, **1995**: Vol. **65**, 93[–]115.

Szatzschneider, Wojciech, “Extended Cox, Ingersoll & Ross Model” *Revista Mexicana de Economía y Finanzas*, **1(4)(2002)**.

Yor, Marc, “On some exponential integral functionals”, *Mathematical Finance* **3(2)(1993):** 125[–]134.

See also: SQUARE ROOT PROCESS/LAW; HULL AND WHITE MODEL; MODELS OF THE TERM STRUCTURE OF INTEREST RATES