Comments about CIR Model

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Introduction

The Cox, Ingersoll & Ross Model for interest rates was proposed in 1980. Since then it has been the objective of many even recent studies and extensions. However, in spite of many deep theoretical studies, there are many errors, imprecisions or trivialities published in the financial literature. The main goal of this article is to correct some of them, and to show how relevant problems concerning CIR are linked to stochastic analysis.

Usually it is assumed risk premia proportional to $\sqrt{r(t)}$; and that linear risk premia are considered inadmissible; cf. Rogers (1995).

However, if one wants to work with CIR Model in Risk Neutral World (RNW) —the only world that can be observed for interest rates (IR) alone—, then it turns out (in some cases) that linear risk premia are allowed. In this case the IR in the Physical (Real) World follow a different model. We solve this problem in section 1 using elementary application of the Girsanov change of measure.

In section 2, we show how to "solve" the problem of pricing bonds in the double square root Longstaff model. The wrong solution was presented by Longstaff (1989) and a simple version was solved by Beaglehole & Tenney (1992).

In the analysis of the original double square root model the local time should appear (omitted by Longstaff). The appearance of local time makes it impossible to find a closed solution. The best we can do is to price bonds maturing at exponential time independent of the process, and use the decomposition of trajectories from, for example, Yor (1994). In section 3, we offer a short discussion of problems that are essentially equivalent or similar to the Longstaff one, with CIR as a short rate:

i) Pricing of options on assets.

ii) Pricing of default bonds in Merton's Structural Approach.

In both cases we assume that Asset Prices follow a Geometric Brownian Motion, and are correlated with IR. The problem *ii*) was "solved" by Wang (1999) assuming independence and once again in terms of Laplace transforms. In section 4 we will comment how to model negative correlations (observed in some financial markets) in the CIR setting. We cannot agree with the statement by Dai & Singleton (2000) that square–root diffusions are theoretically incapable of generating negative correlations.

Therefore from the point of view of applications sections 1 and 4 offers *positive* results and section 2 and 3 rather *negative* ones. Although section 1 is only of theoretical importance only. The full reproduction of formulas in section 3 would be the endless one, so we will quote the "subroutines" from the Borodin & Salminen handbook.

1 Linear Risk Premia

We stress that everything we can say about interest rates is deduced from prices of bonds or other interest rate derivatives, and these are priced in so called Risk Neutral World (RNW). In other words, dealing *only* with interest rates the RW (Real World) can not be observed. Therefore in this case the concept of risk premium is dim.

If one wants to consider the RW for interest rates, this RW must be taken from assets.

We proceed with the construction of the RW for IR (interest rates) such that in the RNW the IR follow the CIR model.

For the CIR model in RW, linear risk premia are inadmissible; cf Cox et al. (1985), Rogers (1995).

We will clarify what can be done and what can not in one dimensional financial market driven by Brownian motion, and asset prices that in the RW (under the law P) follow a Geometric Brownian Motion:

$$dS(t) = S(t)\left[\sigma dW(t) + \mu dt\right].$$

Set (discounted prices) $Z_t = S_t/\beta_t$, where

$$\beta_t = \exp\left(\int_0^t r(s)ds\right),$$

and $r(s)$ is the spot IR in the RW. Now,

$$dZ(t) = Z(t)\left[\sigma dW(t) + (\mu - r(t))dt\right].$$

The RNW is defined as the probability law Q ($Q \sim P$), $t \leq T$ that under Q

$$dZ(t) = \sigma Z(t) dW^*(t),$$

being W^* another Brownian Motion. It can be shown that if $r(t)$ is CIR (in real world) then such Q does not exist. An easy argument is based on explosion until $T = 1$ of the process defined by:

$$dX(t) = dW(t) + X^2(t)dt,$$

cf. Revuz & Yor (1999) p. 384.

But what we really want is CIR in the RNW. We prove the following:
Theorem 1
If under P

$$dr(t) = 2\widetilde{\sigma}\sqrt{r(t)}dW(t) + \left[\delta + 2\frac{\mu}{\sigma}\widetilde{\sigma}\sqrt{r(t)} - \left(2\beta\, r(t) + \frac{2\widetilde{\sigma}}{\sigma}r^{\frac{3}{2}}(t)\right)\right] dt, \qquad (1)$$

then for any $T > 0$, there exists $Q \sim P$, for the process considered until time T such that under Q the interest rates follow:

$$dr(t) = 2\widetilde{\sigma}\sqrt{r(t)}\, dW^*(t) + (\delta - 2\beta\, r(t))\, dt, \ \widetilde{\sigma}, \delta, \beta > 0$$

Proof. Set $\widetilde{\sigma} = 1 = \sigma$.

Because, by elementary calculations, the law $Q = Q^\beta$ is equivalent to the law Q^0 of the corresponding BESQ^δ process, $(\beta = 0)$, and similarly $P = P^\beta \sim P^0$, then it is sufficient to prove the equivalence of P^0 and Q^0. To stress that $\beta = 0$, we change r to X.

Applying Itô-Tanaka to $f(x) = (x)^{\frac{3}{2}}$ and occupation times formulas together with the fact that for BESQ^δ, the local time $L_t^a = 0$, for $a \leq 0$ and $\delta > 0$, we have that under Q^0 the exponential local martingale

$$\exp\left\{-\int_0^t X(s)dW^*(s) - \frac{1}{2}\int_0^t X^2(s)ds\right\} =$$

$$\exp\left\{-\frac{X_t^{\frac{3}{2}}}{3} + \frac{X_0^{\frac{3}{2}}}{3} + \frac{1}{2}(\delta+1)\int_0^t \sqrt{X(s)}ds - \frac{1}{2}\int_0^t X^2(s)ds\right\},$$

is bounded by a constant $k(T)$.

Now it is easy to see that

$$\eta_t = \mathcal{E}\left[\int_0^\cdot (X(s) - \mu)\, dW(s)\right]_t$$

is a true martingale.

Moreover $\eta_t > 0$, P^0 almost everywhere. We conclude that $Q^0 \sim P^0$, and $Q \sim P$ on $\mathcal{F}_T$. A similar proof works if in the RW

$$dS(t) = S(t)\{[(\lambda + 1)r(t) + \mu]dt + \sigma dW(t)\}, \text{ for any } \lambda < 0.$$

Namely, that there exist the corresponding model in RW such that in the RNW the IR follow CIR model. We have just proved that in some cases the linear risk premia for CIR model are admissible, of course in our formulation of the problem.

Remark

If one has "extra degrees of freedom", for example:

$$dS(t) = [\sigma_1 dW_1(t) + \sigma_2 dW_2(t) + \mu dt]\, dS(t) \tag{2}$$

and in the RW $r(t) = r_1(t) \oplus r_2(t)$, independent sum of CIR models driven by W_1 and W_2 respectively, then one can drop the drift for discounted prices. This occurs because

$$\mathcal{E}\left(\int_0^{\cdot} r_1(s) dW_2(s)\right)_t$$

is clearly a true martingale (simply take the conditional expectation).

Therefore one can drop the drift term applying Girsanov Theorem twice. Clearly the incompatibility of CIR in RW and RNW persists.

Assume now that $r_1 \oplus r_2$ can be reduced to a one-factor model. This occurs if $\widetilde{\sigma}_i = \widetilde{\sigma}_2, \beta_1 = \beta_2$ in $dr_i(t) = \widetilde{\sigma}\sqrt{r_i(t)}dW_i(t) + (\delta_i - \beta_i r_i(t))\, dt$ by the Pythagoras Theorem, *cf.* Revuz & Yor(1999).

In this case $dr(t) = \widetilde{\sigma}\sqrt{r(t)}dW^*(t) + (\delta_1 + \delta_2 - \beta r(t))\, dt$, with $\widetilde{\sigma} = \sigma_i, \beta = \beta_i$.

Rewriting (2) as $dS(t) = \left(\sqrt{\sigma_1^2 + \sigma_2^2}dW(t) + \mu dt\right) S(t)$ we conclude that in this case there exists an equivalent martingale measure for discounted prices and CIR model in RW. Here W and $W^*$are correlated in a complicated way.

2 Longstaff Model

In (1989) Longstaff proposed the so–called double square root model defined in Risk Neutral World by:

$$dr(t) = 2\sqrt{r(t)}\, dW(t) + \left(1 - \kappa\sqrt{r(t)} - 2\lambda r(t)\right) dt,\ \kappa, \lambda > 0$$

Note the similarity with (1)!

In this study, for the sake of simplicity, we set $\sigma = 1$ in the original model $\widetilde{r}(t) = \sigma r(t)$. Clearly:

$$\begin{aligned} r(t) &= y^2(t), \text{ where} \\ dy(t) &= dW^*(t) - \left(\lambda y(t) + \frac{\kappa}{2}\operatorname{sgn} y(t)\right) dt. \end{aligned}$$

In 1992 Beaglehole & Tenney showed that Longstaff's *wrong* formula for Bond Prices in his model gives the correct bond prices in the case of:

$$r_1(t) = y_1^2(t), \text{ and } dy_1(t) = dW(t) - \left(\lambda y_1(t) + \frac{\kappa}{2}\right) dt.$$

Longstaff is his study uses Feynman-Kac approach and obtains the formula for bond prices of the form:

$$P(t,T) = exp\{m(t,T) - n(t,T)x - p(t,T)\sqrt{x}\}$$

for some functions m , n, p, and $x = r_t$. However to apply the Feynman-Kac representation, $P(t,T)$ should be of class C^2 with respect to x, *Karatzas* & *Shreve* (1991) theorem 7.6.
Some relaxation of this assumption is imaginable, but there is no possibility to make adjustments that could work for $P(t,T)$. The problem is of course at zero.

We will show first how to calculate:

$$E_x\left(e^{-\int_0^t r_1(s)ds}\right) \tag{3}$$

By very simple and probably known martingale method based on the following:

Let $f(s)$ and $g(s)$ are, for example, differentiable functions, then for any t

$$E_x\left[e^{\int_0^t f(s)W(s)+g(s)dW(s)-\frac{1}{2}\int_0^t (f(s)W(s)+g(s))^2 ds}\right] = 1.$$

The notation E_x means that the process starts at x. In the sequel we shall use the notation $\propto$ for "=" if multiplied by a deterministic function. This function is easy to calculate but we will not reproduce it here. Also for simplicity's sake we choose $x = 0$. The general case is analogous.

By Girsanov Theorem:

$$\begin{aligned}(3) \quad &\propto \quad E\left(e^{-\left(\frac{\lambda^2}{2}+1\right)\int_0^t W^2(s)ds-\frac{\lambda\kappa}{2}\int_0^t W(s)ds-\frac{\lambda}{2}W^2(t)-\frac{\kappa}{2}W(t)}\right)\\ &\propto \quad E\left(e^{\int_0^t (f(s)W(s)+g(s))dW(s)-\frac{1}{2}\int_0^t (f(s)W(s)+g(s))^2 ds}\right)\end{aligned}$$

if and only if in $(0,t)$

$$\begin{aligned} f'(s)+f^2(s) &= \lambda^2+2\\ g(s)f(s)+g'(s) &= \frac{\lambda\kappa}{2}, \text{ and}\\ f(t) &= -\lambda,\\ g(t) &= -\frac{\kappa}{2}.\end{aligned}$$

Therefore the problem of bonds pricing in the Beaglehole & Tenney model is reduced to entirely elementary calculations giving the Longstaff result.

This matching procedure does not work in the original Longstaff model, it means for calculations of:

$$P(0,t) = E_x\left(e^{-\int_0^t r(s)ds}\right).$$

An application of Girsanov theorem leads to

$$P(0,t) \propto E_x\left[e^{-\left(\frac{\lambda^2}{2}+1\right)\int_0^t W^2(s)ds-\frac{\kappa\lambda}{2}\int_0^t |W(s)|ds-\frac{\lambda}{2}W^2(t)-\frac{\kappa}{2}\left(|W(t)|-L_t^0\right)}\right].$$

The positive term $+\kappa L_t^0$ makes it impossible the direct Feynman-Kac approach to the calculation of the Laplace transform of $P(0,t)$ cf Karatzas and Shreve (1991), or equivalently to calculations of $P(0,T)$ where T is an exponential random variable independent of the process.

We shall calculate $P(0,T)$ conditioning with respect to $W(T)$ and L_T as in Yor (1992) proposition 3.2. But in this proposition the process starts at zero, and not at arbitrary x. If one wants to solve the problem reducing first the process to zero, one should know the density of the hitting time of y for the Ornstein-Uhlenbeck process starting at x. This problem has not an explicit solution *cf.* Göing, Yor (2003), (2003).

This problem can be solved considering separately $T < T_0$, $T > T_0$, being T_0 the hitting time of zero. This method would lead to even more complicated expressions. In what follows we assume $r_0 = 0$

Let

$$\widetilde{A}(t) = -\left(\frac{\lambda^2}{2} + 1\right)\int_0^t W^2(s)ds - \frac{\lambda\kappa}{2}\int_0^t |W(s)|\, ds,$$

For calculation of $E_0\left(e^{-\widetilde{A}(\tau)}\right)\left(\tau \sim \exp\frac{\theta^2}{2}\right)$

we refere to general theory where *excursions* enter into the play cf. Yor (1992, 1994), Jeanblanc *et al.* (1996). We shall give some details:

By the proposition 3.2 from Yor (1992),

$$E_0\left(e^{-\widetilde{A}(\tau)} \mid l_\tau = l, W_\tau = a\right) \propto E\left[e^{-\widetilde{A}(\tau_l) - \frac{\theta^2}{2}\tau_l}\right] E_a\left(e^{-\widetilde{A}_{T_0} - \frac{\theta^2}{2}T_0}\right)$$

and τ_l is the inverse local time at zero.

Therefore we have to calculate:

$$\int_0^\infty dl\, e^{\frac{\kappa}{2}l} E_0\left[e^{-\theta^2\frac{\tau_l}{2} - \widetilde{A}(\tau_l)}\right]\int_{-\infty}^\infty E_a\left(e^{-\theta^2\frac{T_0}{2} - \widetilde{A}(T_0)}\right)e^{-\frac{\lambda}{2}a^2 - \frac{\kappa}{2}|a|}da.$$

Calculations of $E_a\left(e^{-\theta^2\frac{T_0}{2} - \widetilde{A}(T_0)}\right)$ can be reduced to the formula 2.0.1 page 429 from Borodin & Salminen. This reduction we will present in the appendix. On the other hand this formula represents the solution of the equation:

$$\frac{1}{2}v''(a) = \left(\frac{\theta^2}{2} + f(a)\right)v(a), 0 \le v(a) \le 1\ ,\ v(0) = 1, \tag{4}$$

where

$$f(a) = \left(\frac{\lambda^2}{2} + 1\right)a^2 + \frac{\lambda\kappa}{2}|a|\,.$$

In the final part we will need another solution $h(m)$ of the equation (4)

$$h(m) = v(m)\int_0^m \frac{1}{v^2(s)}ds, \tag{5}$$

cf. Jeanblanc et al. (1996). Define $u(m) = \frac{h(m)}{m}$.

Elementary calculations show that $u(0)=1,\ u(m)>1$, for $m>0$

The final part (the most interesting from the point of view of stochastic analysis) is the calculation of

$$\int_0^\infty dl\, e^{\frac{\kappa l}{2}} E_0\left(e^{-\theta^2\frac{\tau_l}{2}-\widetilde{A}(\tau_l)}\right)$$

(note that of course the integral does exist). There are two possibilities of calculations of this integral:

i) Calculations in terms of Ray-Knight theorem.

By occupation time and Ray-Knight theorems, cf. Revuz & Yor (1999), we have

$$\begin{aligned}\int_0^{\tau_l} ds f(W(s)) &= \int_{-\infty}^{+\infty} f(x) L^x_{\tau_l} dx \\ &= \int_{-\infty}^{+\infty} \left(\left(\frac{\lambda^2}{2}+1\right)x^2+\frac{\lambda\kappa}{2}|x|\right) L^x_{\tau_l} dx \\ &= \int_0^{\infty} \left(\left(\frac{\lambda^2}{2}+1\right)x^2+\frac{\lambda\kappa}{2}x\right)\left(X_1(x)+X_2(x)\right) dx \\ &= \int_0^{+\infty} g(x)\left(X_1(x)+X_2(x)\right) dx\end{aligned}$$

where $X_1,\ X_2$ are two independent squared Bessel processes of dimension zero starting at l. Putting $\frac{\theta^2\tau_l}{2}$ inside the integral and applying Pitman & Yor formula for squared Bessel processes we have that:

$$E_0\left(e^{-\frac{\theta^2\tau_l}{2}-\widetilde{A}(\tau_l)}\right)=e^{lv^+(0)},$$

being v^+ right hand derivative at zero, of the function v defined by formula (4).

Therefore

$$\int_0^\infty \exp\left(\frac{\kappa l}{2}\right)\exp\left(lv^+(0)\right) dl=\frac{1}{-v^+(0)-\frac{\kappa}{2}}.$$

ii) The second way of calculations is given in terms of the excursion theory and this will lead to more explicite formula. We follow closely the general approach from Yor (1994). Results easily from the multiplicative formula for excursions that

$$\frac{\theta^2}{2}\int e^{\kappa l}E_0\left(e^{-\theta^2\frac{\tau_l}{2}-\widetilde{A}(\tau_l)}\right) dl=\frac{\theta^2}{2(D_\theta-\frac{\kappa}{2})},$$

where

$$D_\theta(f)=\int \mathbf{n}(d\varepsilon)\left[1-e^{-\frac{\theta^2}{2}V-\int_0^V ds f(\varepsilon_s)}\right]$$

cf. Yor (1994) pages 69 and 75.

We know a priori that $D_\theta > \frac{\kappa}{2}$.

Now D_θ is given by:

$$\int_0^\infty \frac{dm}{m^2}\left(1-\frac{1}{u(m)}\right)^2.$$

This, because of Williams representation of excursions, $\frac{1}{m^2}$ is the 'law' of the maximum and conditioning we have to calculate

$$E_0^{(3)}\left(e^{-\frac{\theta^2}{2}T_m-\int_0^{T_m}dt\,f(R(t))}\right), \text{where}$$

$R(t)$ is BES^3 process starting at zero, T_m is the hitting time of m, and therefore $u(m)$ is given by (5), *cf* Yor(1994).

3 Related Problems

In this section we review briefly another problems concerning CIR once again of theoretical value only. These problems are closely related to the Longstaff model.

a. *Default bonds in the structural Merton approach.* For discussion we refer to the paper by Wang(1999), who solved the problem in the case of:

   i. Default occuring at the time T (the horizon).

   ii. The value on the firm follows geometric Brownian motion *independent* of the CIR interest rates.

   In our approach we do not assume independence. We solve the problem in this setting if we know how to price options on assets with CIR as a short rate.

b. *Options on assets with CIR as a short rate.* Assume that an asset follows geometric Brownian motion driven by $W(t)$, and interest rates follow $r_1(t) \oplus r_2(t)$, where $\oplus$ stands for the independent sum.

Here $r_2(t)$ is CIR, and $r_1(t)$ is one dimensional CIR model driven by $W(t)$.

The analytical solution of pricing options is equivalent to the knowledge of the joint law of $\int_0^t r_1(s)\,ds$ and $W(t)$. To calculate the Laplace transform of

$$E\left(e^{-\lambda\int_0^t r_1(s)ds+\mu W(t)}\right)$$

we use Girsanov's theorem and the problem is equivalent to pricing bonds in the Longstaff model. Note that even in Wang's case, one has to invert the corresponding Laplace transform! Of course double inversion (in our case) of Laplace transform rules off applications.

4 Negative correlations

In this section we will explain briefly how to obtain negative correlations between factors in CIR framework.

The problem is extremely important and could lead toward future applications. In this first approach we will consider only two factors. As mentioned in Dai & Singleton (2000), "The data on U.S. interest rates seems to call for negative correlations among the risk factors. Because CSR (Correlated Square Root) models are theoretically incapable of generating negative correlations, we conclude that they are not consistent with the historical behaviour of U.S. interest rates". Of course CIR models driven by *the same* Brownian motion cannot be negatively correlated. Therefore we have to relax assumption "driven by the same Brownian motion".

A natural form to obtain a negative correlation could be by setting:

$$\begin{array}{rcl} r_1(t) & \rightarrow & \text{CIR driven by } W(t) \\ r_2(t) & \rightarrow & \text{CIR driven by } -W(t) \end{array}$$

The problem is that there is no easy approach to calculate

$$E\left(\exp - \int_0^t (r_1(s) + r_2(s))ds\right).$$

Similar comment was made by Schönbucher (2003), p 175. Further we read "Alternatively one could restrict the specification to a squared Gaussian model". Our following construction represents a generalization of this proposal.

Set

$$r_1(t) = \hat{r}_1(t) + \alpha(W(t) + A)^2$$
$$r_2(t) = \hat{r}_2(t) + \beta(W(t) - B)^2, \quad \text{for}$$

$\alpha, \beta, A, B > 0$, and A and B *Large enough.* Here $\hat{r}_1(t)$, $\hat{r}_2(t)$ are CIR models, and $\hat{r}_1$, $\hat{r}_2$, W are independent processes ($W(t)$ is Brownian Motion)

$$Cov(r_1(t), r_2(t)) = 2\alpha\beta t(t - 2AB).$$

Note that $(W(t) + A)^2$ and $(W(t) - B)^2$ are one dimensional squared Bessel processes driven by

$$\begin{array}{rcl} B_1(t) & = & \int_0^t sgn(W(s) + A)dW(s), \quad \text{and} \\ B_2(t) & = & \int_0^t sgn(W(s) - B)dW(s). \end{array}$$

$E(\exp - \int_0^t r_i(s)ds)$, for $i = 1, 2$ and $E(\exp - \int_0^t (r_1(s) + r_2(s))ds$ can be calculated easily. A similar procedure (using multidimensional Brownian motion) can be applied for more factors.

Finally, one could be tempted to set:

$$\begin{aligned} r_1(s) &= \hat{r}_1(s) + \alpha W^2(s) 1(W(s) > 0) \\ r_2(s) &= \hat{r}_2(s) + \alpha W^2(s) 1(W(s) < 0) \end{aligned}$$

However, calculations of $E \exp - \int_0^t r_1(s)ds$ are similar to pricing bonds in Longstaff model, as explained in Pitman & Yor (1982) and Yor (1994)!

5 Appendix

We will give here the sketch of how to calculate:

$$E_a \left(e^{-\theta^2 \frac{T_0}{2} - \tilde{A}(T_0)} \right) =$$

$$E_a \left(e^{-\frac{\lambda^2+2}{2}} \int_0^{T_0} W^2(s)ds - \frac{\lambda\kappa}{2} \int_0^{T_0} W(s)ds - \theta^2 \frac{T_0}{2} \right) \tag{6}$$

set $b = \frac{\lambda\kappa}{2(\lambda^2+2)}$
(6) is equal to

$$E_{a+b} \left(e^{-\frac{\lambda^2+2}{2}} \int_0^{T_b} W^2(s)ds - \left(\frac{\theta^2}{2} + \frac{\lambda\kappa}{2(\lambda^2+2)} \right) T_b \right) \tag{7}$$

T_b is the hitting time of b, and Brownian Motion starts at $a+b$.
Simplifying

$$(7) = E_{a+b} \left(e^{-\frac{c^2}{2} \int_0^{T_b} W^2(s)ds - \frac{\tilde{\theta}^2}{2} T_b} \right)$$

$$\tilde{E}_{a+b} \left(e^{-\frac{cW^2(T_b)}{2} - \frac{c+\tilde{\theta}^2}{2} T_b} \right)$$

$\tilde{E}$ stands for the expectation with respect to the corresponding law of Orstein-Uhlenbeck process, and the change of measure is through Girsanov theorem. Because $W^2(T_b) = b^2$ we reduce the problem to the calculation of $\widetilde{E}_{a+b}\left(e^{T_b}\right)$ and precisely this formula appears in the Borodin & Salminen textbook where complicated parabolic cylinder functions appear *cf.* p 450.

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